

Free Vibration Analysis of Timoshenko Beam Using Energy Separation Principle

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Abstract

This paper proposes a model for the free vibration analysis of Timoshenko beam in which the finite element method is applied in conjunction with the energy method. The Timoshenko beam is divided into two virtual beams, namely Euler-Bernoulli beam and shear layer beam. The proposed analytical relationship between bending and shear rotations of the Euler-Bernoulli beam is established through the use of a bending-shear rotation interdependent factor.

The results show high accuracy and efficiency of the proposed element in calculating the natural vibration frequencies of beams with different support conditions. It is concluded that the effect of shear deformation is significant in beams whose span-to-depth ratio is less than 5, and should therefore be accounted for in their design.

Keywords: Timoshenko beam, Free vibration, Finite element, Shear deformation, Rotary inertia, Shear locking, Natural frequency

1 INTRODUCTION

The failure of major structures such as bridges, high-rise buildings and airplane wings is an awesome possibility under resonance. Thus the calculation of element natural frequencies is of major importance in the study of vibration of beams. The Euler Bernoulli beam theory, sometimes called the classical beam theory, is the most commonly used because it is simple and provides reasonable engineering approximations for many problems. Doyle and Pavlovic [1] and Gurg'oze [2, 16] obtained the frequency equation of a clamped-free Bernoulli-Euler beam with attached tip mass and a spring-mass system by using the Lagrange multipliers method. However, the Euler-Bernoulli model tends to slightly overestimate the natural frequencies, especially the natural frequencies of higher modes.

The extension of Euler-Bernoulli beam theory is the inclusion of the effect of rotary inertia, known as Rayleigh's theory. Early investigators such as Bapat and Bapat [3] investigated the natural frequencies of an Euler beam with concentrated masses. Other researchers such as Chang [4] solved a simply-supported Rayleigh beam carrying a rigidly-attached centred mass. It partially corrects the overestimation of natural frequencies in the Euler-Bernoulli model.

Timoshenko's theory includes the shear effect on the vibration of beam. The superiority of the Timoshenko model is more pronounced for beams with a low aspect ratio. Due to the complexity of the governing equations of free vibrations of beams in general, numerical methods

such as finite element methods have been developed profoundly. The major issue with finite element formulations for Timoshenko beams is that when span-to-depth ratio of the beam is low strong stiffening of the elements occurs, resulting in spurious shear stress predictions and erroneous results for the generalized displacements, [5, 17]. This phenomenon is known as shear-locking.

The usual engineering practice to neglect the secondary effects such as rotary inertia and transverse shear in calculating the natural frequencies may be justified to some extent for slender beams, at best for few first modes. In this case, the influence of the secondary effects is small. However, for stocky beams the secondary effects become more important, especially for higher modes.

In this paper, a unified beam element model of the Euler-Bernoulli and Timoshenko beam theories is proposed for the free vibration analysis of Timoshenko beams.

2 FORMULATION OF BENDING-SHEAR INTERACTION FACTOR

In formulation of the interpolation function, the beam deflection w is divided into two components; that due to the flexure, w_b , and that due to transverse shear, w_s . The angle of rotation of the cross-section θ is divided into its constitutive parts; the angle of cross-section rotation due to bending, θ_b and the cross-section slope due to shear, θ_s (see Figure 1b).

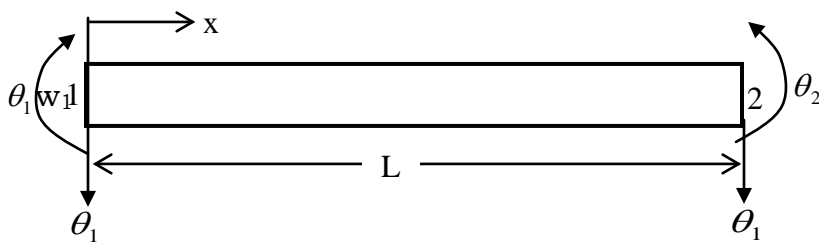


Figure 1a – Beam element

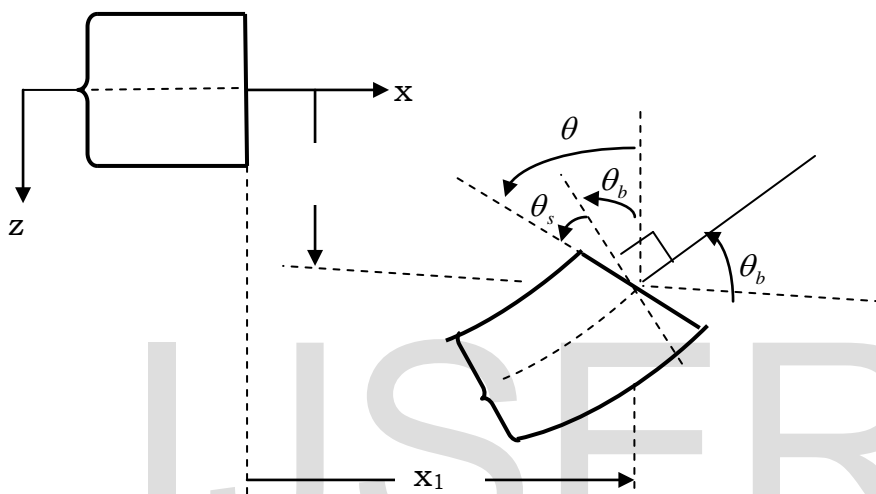


Figure 1b – Kinematics of a beam undergoing both bending and shear rotations

To ensure continuous interaction between the bending and shear components as a function and avoid the use of partial derivatives, the following relationship for the total cross sectional rotation θ is proposed, [6]:

$$\theta(x) = \beta\theta_b(x) + (1 - \beta)\theta_s(x) \tag{1}$$

where $\theta(x)$ is the total cross-sectional rotation of the beam

$\theta_b(x)$ is the cross-sectional rotation of the Euler-Bernoulli beam

$\theta_s(x)$ is the cross-sectional rotation of the shear beam

β is the bending-shear interaction factor and is expressed as the ratio of bending strain energy to total strain energy of a simply-supported beam under load.

That is:

$$\beta = \frac{q_b}{q} = \frac{U_b}{U_b + U_s} = \frac{1}{1 + \Phi} \tag{2}$$

where $\Phi = \frac{U_s}{U_b}$

U_b = strain energy in bending deformation

U_s = strain energy due to shear deformation

The integral expression for bending strain energy is given by the familiar expression:

$$U_b = \int_0^L \frac{(M(x))^2}{2EI} dx \tag{3}$$

where E is the elastic modulus of the beam material.

I = moment of inertia of the beam section.

Consider a simply supported beam with a point load P at midspan.

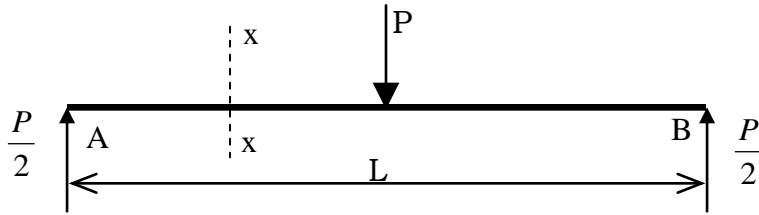


Figure 2- A simply supported beam under a point load P at the center

The bending moment at a section, distance x from a support, is given by:

$$M(x) = \frac{Px}{2}, x < L/2 \quad \text{and} \quad M(x) = \frac{Px}{2} - P\left(x - \frac{L}{2}\right), x > L/2 \quad (4)$$

Since the maximum bending moment occurs at midspan ($x=L/2$),

$$M(x) = \frac{Px}{2}$$

Substituting for $M(x)$ in Equation (3) and performing integration gives

$$\therefore U_b = \frac{P^2 L^3}{96EI} \quad (5)$$

The shear force at any section, distance x from a support, is:

$$Q(x) = \frac{P}{2} \quad (6)$$

The integral expression for shear strain energy is given by the familiar expression

3 FORMULATION OF INTERPOLATION FUNCTIONS

Hermite cubic polynomial is used to approximate the flexural deformation, $w_b(x)$: [6, 7]

$$w_b(x) = a_1 + a_2x + a_3x^2 + a_4x^3 \quad (10)$$

$$U_s = \int_0^L \frac{Q(x)^2}{2kAG} dx \quad (7)$$

Substituting for $Q(x)$ in Equation (7) gives the shear strain energy as:

$$U_s = \frac{P^2 L}{8kAG} \quad (8)$$

$$\therefore \Phi = \frac{U_s}{U_b} = \frac{12EI}{L^2 kAG} \quad (9)$$

where E = Young's modulus

G = shear modulus

A = cross-sectional area

k = shear coefficient depending on the shape of cross-section.

Edem [5] proposed that the bending-shear interaction factor, β , be based on the value of Φ for midspan point load, i.e. Equation (9).

$$\theta_b(x) = \frac{dw_b(x)}{dx} = a_2 + 2a_3x + 3a_4x^2$$

The generalized nodal displacements for the Bernoulli beam are defined as w_b and θ_b .

$$w_b(x) = \sum_{i=1}^4 \varphi_i(x) u_i \quad (11)$$

where φ_i 's are given as

$$\varphi_1 = \frac{L^3 - 3Lx^2 + 2x^3}{L^3}, \quad \varphi_2 = \frac{xL^3 - 2L^2x^2 + Lx^3}{L^3}, \quad \varphi_3 = \frac{3Lx^2 - 2x^3}{L^3} \quad (12)$$

and $\{u\}$ denotes the column displacement vectors

$$\{w_{b1}, \theta_{b1}, w_{b2}, \theta_{b2}\}^T$$

The Interpolation Function for shear deformation (w_s) is approximated using a quadratic polynomial: [6, 7]

$$w_s(x) = 1 + b_1 + b_2x + b_3x^2 \quad (13)$$

$$\theta_s(x) = \frac{dw_s(x)}{dx} = 1 + b_2 + 2b_3x$$

The generalized nodal displacements for the shear beam are defined as w_s and θ_s .

$$w_s(x) = \sum_{i=1}^4 \eta_i(x) u_i \quad (14)$$

where η_i 's are given as

$$\eta_1 = 1 - \frac{x}{L}, \eta_2 = \frac{L}{2} \left(\frac{x}{L} - \left(\frac{x}{L} \right)^2 \right), \eta_3 = \frac{x}{L} \text{ and } \eta_4 = -\frac{L}{2} \left(\frac{x}{L} - \left(\frac{x}{L} \right)^2 \right) \quad (15)$$

and $\{u\}$ denotes the column displacement vectors $\{w_{s1}, \theta_{s1}, w_{s2}, \theta_{s2}\}^T$

4 FORMULATION OF TIMOSHENKO BEAM ELEMENT STIFFNESS MATRIX

The expression for strain energy in the proposed unified beam element is obtained by integrating the expression for strain energy per unit length of the beam.

The total energy in the unified beam element under a distributed normal load q is in form of: [8, 9]

$$U(\theta_b, \theta_s) = \frac{EI}{2} \int_0^L [\beta \dot{\theta}_b^2 + (1 - \beta) \dot{\theta}_s^2] dx \quad (16)$$

$$K_{ij} = \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} \frac{1}{2} \int_0^L \beta EI \left(\sum_{k=1}^4 \ddot{\varphi}_k(x) u_k \right)^2 dx + \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} \frac{1}{2} \int_0^L (1 - \beta) EI \left(\sum_{k=1}^4 \ddot{\eta}_k(x) u_k \right)^2 dx$$

$$= \int_0^L \beta EI \ddot{\varphi}_i(x) \ddot{\varphi}_j(x) dx + \int_0^L (1 - \beta) EI \ddot{\eta}_i(x) \ddot{\eta}_j(x) dx \quad (20)$$

where $\ddot{\varphi}_i = \frac{d^2 \varphi_i}{dx^2}$ and $\ddot{\eta}_i = \frac{d^2 \eta_i}{dx^2}$

The assembled unified beam element stiffness matrix K is

$$\dot{\theta}_b(x) = \sum_{i=1}^4 \dot{\varphi}_i(x) u_i \quad \text{for the flexural beam} \quad (17)$$

$$\dot{\theta}_s(x) = \sum_{i=1}^4 \dot{\eta}_i(x) u_i \quad \text{for the shear beam} \quad (18)$$

From Castigliano's first theorem, the stiffness coefficient K_{ij} is given by: [18, 20]

$$K_{ij} = \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} U_p(\theta) \quad (19)$$

$$K = \frac{\beta EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4 + \Phi)L^2 & -6L & (2 - \Phi)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & (2 - \Phi)L^2 & -6L & (4 + \Phi)L^2 \end{bmatrix} \quad (21)$$

$$\Phi = \frac{1 - \beta}{\beta}$$

5 BEAM ELEMENT MASS MATRIX

The kinetic energy of a particle of density ρ and cross sectional area A within a beam element moving with velocity, v , is given by: [9, 19]

$$U_k^i = \frac{1}{2} \rho A v^2 \quad (22)$$

The element velocity field can be approximated by the shape functions: [10, 11]

$$v = \frac{d(\beta w_b(x) + (1 - \beta)w_s(x))}{dt} = \frac{d\left(\beta \sum_{i=1}^4 \varphi_i(x)u_i + (1 - \beta) \sum_{i=1}^4 \eta_i(x)u_i\right)}{dt} \quad (24)$$

For the beam element, the total kinetic energy U_k is

$$U_k = \frac{1}{2} \int_0^L \rho A v^2 dx \quad (23)$$

Substituting Equation (24) into Equation (23):

$$U_k = \frac{1}{2} \int_0^L \rho A \left(\beta \sum_{i=1}^4 \varphi_i(x)u_i \right)^2 dx + \frac{1}{2} \int_0^L \rho A \left((1 - \beta) \sum_{i=1}^4 \eta_i(x)u_i \right)^2 dx \quad (25)$$

The mass coefficient M_{ij} is given by [9, 19]

$$M_{ij} = \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} U_k$$

$$= \frac{\partial^2}{\partial u_i \partial u_j} \frac{1}{2} \int_0^L \rho A \left(\beta \sum_{i=1}^4 \varphi_i(x)u_i \right)^2 dx + \frac{\partial^2}{\partial u_i \partial u_j} \frac{1}{2} \int_0^L \rho A \left((1 - \beta) \sum_{i=1}^4 \eta_i(x)u_i \right)^2 dx$$

i.e. $M_{ij} = \beta \int_0^L \rho A \varphi_i(x) \varphi_j(x) dx + (1 - \beta) \int_0^L \rho A \eta_i(x) \eta_j(x) dx$ (26)

where $\int_0^L \varphi_{ij} dx = \int_0^L \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & \varphi_{14} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & \varphi_{24} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} & \varphi_{34} \\ \varphi_{41} & \varphi_{42} & \varphi_{43} & \varphi_{44} \end{bmatrix} dx$

and $\int_0^L \eta_{ij} dx = \int_0^L \begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & \eta_{23} & \eta_{24} \\ \eta_{31} & \eta_{32} & \eta_{33} & \eta_{34} \\ \eta_{41} & \eta_{42} & \eta_{43} & \eta_{44} \end{bmatrix} dx$

where the φ_i 's and η_i 's are given by Equations (12) and (15) and

$$\varphi_{11} = \varphi_1 \times \varphi_1, \varphi_{12} = \varphi_1 \times \varphi_2, \text{ etc}$$

$$\eta_{11} = \eta_1 \times \eta_1, \eta_{12} = \eta_1 \times \eta_2, \text{ etc}$$

$$M_b = \frac{\beta \rho A L}{840} \begin{bmatrix} (312 + 280\Phi) & (44 + 35\Phi)L & (108 + 140\Phi) & -(26 + 35\Phi)L \\ (44 + 35\Phi)L & (8 + 7\Phi)L^2 & (26 + 35\Phi)L & -(6 + 7\Phi)L^2 \\ (108 + 140\Phi) & (26 + 35\Phi)L & (312 + 280\Phi) & -(44 + 35\Phi)L \\ -(26 + 35\Phi)L & -(6 + 7\Phi)L^2 & -(44 + 35\Phi)L & (8 + 7\Phi)L^2 \end{bmatrix} \quad (27)$$

where $\Phi = \frac{1 - \beta}{\beta}$

6 BEAM ELEMENT ROTARY INERTIA MATRIX

The total kinetic energy due to rotation of the cross-section during bending is given by: [19]

$$U_{ki} = \frac{1}{2} \int_0^L \rho I \theta^2 dx \quad (28)$$

But the rotation of the cross-section is given by:

$$\theta = \beta \theta_b + (1 - \beta) \theta_s(x) = \beta \sum_{i=1}^4 \dot{\varphi}_i u_i + (1 - \beta) \sum_{i=1}^4 \dot{\eta}_i(\bar{x}) u_i \quad (29)$$

Substituting for θ in Equation (28) gives:

$$U_{ki} = \frac{1}{2} \int_0^L \rho I \left(\beta \sum_{k=1}^4 \dot{\varphi}_i(x) u_i \right)^2 dx + \frac{1}{2} \int_0^L \rho I \left((1 - \beta) \sum_{i=1}^4 \dot{\eta}_i(x) u_i \right)^2 dx \quad (30)$$

The rotary inertia coefficient R_{ij} is given by: [18, 21]

$$R_{ij} = \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} U_{ki}$$

$$= \frac{\partial^2}{\partial u_i \partial u_j} \frac{1}{2} \int_0^L \rho I \left(\beta \sum_{k=1}^4 \dot{\varphi}_i(x) u_i \right)^2 dx + \frac{\partial^2}{\partial u_i \partial u_j} \frac{1}{2} \int_0^L \rho I \left((1 - \beta) \sum_{i=1}^4 \dot{\eta}_i(x) u_i \right)^2 dx$$

$$i.e. R_{ij} = \beta \int_0^L \rho I \dot{\varphi}_i(x) \dot{\varphi}_j(x) dx + (1 - \beta) \int_0^L \rho I \dot{\eta}_i(x) \dot{\eta}_j(x) dx \quad (31)$$

where $\dot{\varphi}_i = \frac{d\varphi_i}{dx}$ and $\dot{\eta}_i = \frac{d\eta_i}{dx}$

$$\text{and } \int_0^L \dot{\varphi}_{ij} dx = \int_0^L \begin{bmatrix} \dot{\varphi}_{11} & \dot{\varphi}_{12} & \dot{\varphi}_{13} & \dot{\varphi}_{14} \\ \dot{\varphi}_{21} & \dot{\varphi}_{22} & \dot{\varphi}_{23} & \dot{\varphi}_{24} \\ \dot{\varphi}_{31} & \dot{\varphi}_{32} & \dot{\varphi}_{33} & \dot{\varphi}_{34} \\ \dot{\varphi}_{41} & \dot{\varphi}_{42} & \dot{\varphi}_{43} & \dot{\varphi}_{44} \end{bmatrix} dx$$

$$\dot{\phi}_{11} = \dot{\phi}_1 \times \dot{\phi}_1, \dot{\phi}_{12} = \dot{\phi}_1 \times \dot{\phi}_2, \text{ etc}$$

$$R = \frac{\beta \rho I}{30L} \begin{bmatrix} (36 + 30\Phi) & 3L & -(36 + 30\Phi) & 3L \\ 3L & (4 + \frac{5}{2}\Phi)L^2 & -3L & -(1 + \frac{5}{2}\Phi)L^2 \\ -(36 + 30\Phi) & -3L & (36 + 30\Phi) & -3L \\ 3L & -(1 + \frac{5}{2}\Phi)L^2 & -3L & (4 + \frac{5}{2}\Phi)L^2 \end{bmatrix} \quad (32)$$

7 GOVERNING EQUILIBRIUM EQUATIONS

Consider a beam of uniform cross section made of homogenous isotropic material.

The governing equation of dynamic equilibrium for an undamped structure is:

$$[M]\{\ddot{U}\} + [K]\{U\} = \{F_{ext}\} \quad (33)$$

where [M] is the Structure mass matrix
 [K] is the Structure stiffness matrix
 {U} is the vector of the structural nodal displacements
 {F_{ext}} is the vector of nodal external forces

$\ddot{U} = \frac{d^2U}{dt^2}$ is the acceleration of the material particles of the structure

In free vibration, there is no external force and damping is zero.

Therefore Equation (33) becomes:

$$[M]\{\ddot{U}\} + [K]\{U\} = \{0\} \quad (34)$$

Substituting $\int_0^L \dot{\phi}_{ij} dx$ and $\int_0^L \dot{\eta}_{ij} dx$ into Equation (31), the beam element rotary inertia matrix is shown to be:

It may be safely assumed that the free vibration motion is simple harmonic. [12, 22]

$$\text{i.e. } \{U\} = \{\hat{U}\} \sin(\omega t + \theta) \quad (35)$$

{Ū} represents the amplitude of vibration, ω is the circular frequency and θ is the phase angle.

Therefore Equation (34) becomes

$$([K] - \omega^2 [M])\{\hat{U}\} = \{0\} \quad (36)$$

Equation (36) is an eigenvalue equation which gives nontrivial solution if the determinant of the coefficients of {Ū} equals zero, i.e.

$$\det([K] - \omega^2 [M]) = \{0\} \quad (37)$$

The eigenvalues ($\omega_1^2, \omega_2^2, \omega_3^2, \dots, \omega_N^2$) represent the frequencies of the N modes of vibration which are possible in the system.

The structure mass matrix [M] in Equation (37) is equal to the sum of matrices due to element mass and rotary inertia. Substituting for [K] and [M] in Equation (37):

$$\det \begin{bmatrix} \frac{\beta EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4L^2 + \Phi L^2) & -6L & (2L^2 - \Phi L^2) \\ -12 & -6L & 12 & -6L \\ 6L & (2L^2 - \Phi L^2) & -6L & (4L^2 + \Phi L^2) \end{bmatrix} \\ - \frac{\beta \rho A L}{840} \begin{bmatrix} (312 + 280\Phi) & (44 + 35\Phi)L & (108 + 140\Phi) & -(26 + 35\Phi)L \\ (44 + 35\Phi)L & (8 + 7\Phi)L^2 & (26 + 35\Phi)L & -(6 + 7\Phi)L^2 \\ (108 + 140\Phi) & (26 + 35\Phi)L & (312 + 280\Phi) & -(44 + 35\Phi)L \\ -(26 + 35\Phi)L & -(6 + 7\Phi)L^2 & -(44 + 35\Phi)L & (8 + 7\Phi)L^2 \end{bmatrix} \\ - \frac{\beta \rho I}{30L} \begin{bmatrix} (36 + 30\Phi) & 3L & -(36 + 30\Phi) & 3L \\ 3L & (4 + \frac{5}{2}\Phi)L^2 & -3L & -(1 + \frac{5}{2}\Phi)L^2 \\ -(36 + 30\Phi) & -3L & (36 + 30\Phi) & -3L \\ 3L & -(1 + \frac{5}{2}\Phi)L^2 & -3L & (4 + \frac{5}{2}\Phi)L^2 \end{bmatrix} \end{bmatrix} = \{0\} \quad (38)$$

8 RESULTS AND DISCUSSION

Consider the rectangular clamped-free beam problem by solved by Reddy [13, 14]

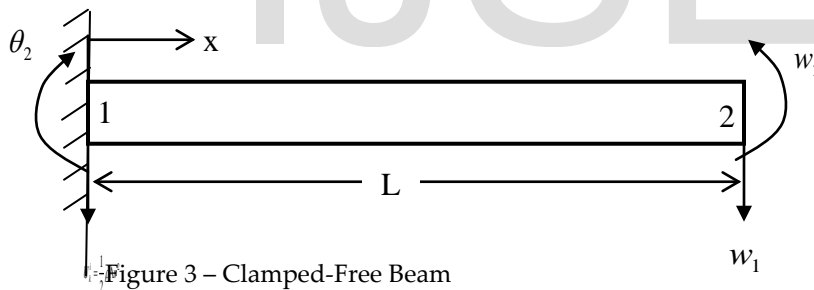


Figure 3 – Clamped-Free Beam

Assumed parameters: Poisson's ratio, $\nu=0.25$; shear coefficient, $k=5/6$

The boundary conditions are $w_1 = \theta_1 = 0$

Solving Equation (38) for a one-element mesh gives:

$$\omega = \sqrt{12.4802 + 9.3240\Phi + (69.6069 + 58.8483\Phi)\lambda} \left(L^2 \sqrt{\frac{EI}{\rho A}} \right) \quad (39)$$

$$\text{where } \lambda = \frac{7}{3} \left(\frac{d}{L} \right)^2 \quad (40)$$

$$\Phi = \frac{12EI}{L^2 GkA} = 3 \left(\frac{d}{L} \right)^2 \quad (41)$$

$$\text{(a) For } L/d=100 \text{ (slender beam), } \omega = 3.52781L^2 \sqrt{\frac{EI}{\rho A}} \quad (42)$$

$$\text{(b) For } L/d=10 \text{ (deep beam), } \omega = 3.5667L^2 \sqrt{\frac{EI}{\rho A}} \quad (43)$$

Neglecting the effect of rotary inertia, Equation (39) reduces to:

$$\omega = \sqrt{12.4802 + 9.3240\Phi} \left(L^2 \sqrt{\frac{EI}{\rho A}} \right) \quad (44)$$

Substituting $\Phi = 3 \left(\frac{d}{L} \right)^2$ into Equation (44):

(a) For L/d=100 (slender beam), $\omega = 3.5331L^2 \sqrt{\frac{EI}{\rho A}}$ (45)

(b) For L/d=10 (deep beam), $\omega = 3.5721L^2 \sqrt{\frac{EI}{\rho A}}$ (46)

The summary of results for different support conditions is shown in Table1 and Table 2.

TABLE 1
DERIVED NATURAL FREQUENCY FORMULAE FOR VARIOUS SUPPORT CONDITIONS

Support Condition	Natural Frequency in rad/s (x $L^2 \sqrt{\frac{EI}{\rho A}}$)		Frequency Ratio $C_\omega = \frac{\omega}{\omega_b}$
	Unified Beam, ω	Timoshenko Beam, ω_b (Exact)**	
Clamped-Free	$\sqrt{12.4802 + 9.3240\Phi + (69.6069 + 58.8483\Phi)\lambda}$	3.5158	$\sqrt{1.01 + 0.75\Phi + (5.63 + 4.76\Phi)\lambda}$
Hinged-Hinged	$\sqrt{(1 + \Phi)(120 + 1680\lambda)}$	9.8776	$\sqrt{(1 + \Phi)(1.2299 + 17.2191\lambda)}$
Clamped-Hinged	$\sqrt{242.1391 + 164.9525\Phi + (2319.89 + 11003.76\Phi)\lambda}$	15.420	$\sqrt{1.02 + 0.69\Phi + (9.75 + 46.28\Phi)\lambda}$
Clamped-Clamped	$\sqrt{516.9321 + 108.9632\Phi + (4480 + 26880\Phi)\lambda}$	22.373	$\sqrt{1.03 + 0.22\Phi + (8.95 + 53.70\Phi)\lambda}$

Legend:

ρ = Density of material (kg/m³), I = Moment of Inertia of cross-section (m⁴), A = Area of Cross-section (m²), E = Modulus of elasticity (N/m²), L = Length of beam (m)

ω = frequency from unified element solution

ω_b = frequency from exact (or classical solution)

** Reddy [13] obtained exact solutions for Timoshenko beam for L/d=100, including the effect of rotary inertia

Note that the frequency ratio, C_ω , is defined as the ratio of the frequency obtained from the unified element solution to that obtained from the exact (or classical) solution.

TABLE 2
COMPUTED NATURAL FREQUENCIES FOR VARIOUS SUPPORT CONDITIONS

Span/depth (L/d) Ratio	Natural Frequency in rad/s (x L ² √(EI/ρA))							
	Clamped-Free		Hinged-Hinged		Clamped-Hinged		Clamped-Clamped	
	With RI	Without RI	With RI	Without RI	With RI	Without RI	With RI	Without RI
10 (Deep beam)	3.3312	3.5721	9.1259	11.1176	13.6141	15.7190	19.9256	22.8079
100 (Slender beam)	3.5308	3.5331	10.9382	10.9561	15.5450	15.5624	22.7139	22.7369

Legend:

RI = Rotary Inertia, ρ = Density of material (kg/m³), I = Moment of Inertia of cross-section (m⁴), A = Area of Cross-section (m²), E = Modulus of elasticity (N/m²), L = Length of beam (m)

The unified element solutions are compared with the exact solutions provided by Reddy [13].

Assumed parameters: Poisson's ratio, ν=0.25; shear coefficient, k=5/6, Span-to-depth (L/d) ratio = 100

$$\text{Therefore } \lambda = \frac{7}{3} \left(\frac{d}{L} \right)^2, \quad \Phi = \frac{12EI}{L^2 GkA} = 3 \left(\frac{d}{L} \right)^2$$

The results are presented in Table 3.

TABLE 3
COMPARISON OF UNIFIED ELEMENT SOLUTION WITH EXACT SOLUTION FOR VARIOUS SUPPORT CONDITIONS

Support Condition	Natural Frequency, ω, in rad/s (x L ² √(EI/ρA))		% Difference
	Unified Beam Solution	Timoshenko Beam (Exact Solution)	
Clamped-Free	3.5308	3.5158	0.42
Hinged-Hinged	9.9392	9.8776	0.62
Clamped-Hinged	15.5450	15.420	0.80
Clamped-Clamped	22.7139	22.373	1.50

Legend:

ρ = Density of material (kg/m³), I = Moment of Inertia of cross-section (m⁴), A = Area of Cross-section (m²), E = Modulus of elasticity (N/m²), L = Length of beam (m)

The relationship between the span-to-depth (L/d) ratio and the natural frequency ratio, C_ω, is presented in Table 4.

$$\lambda = \frac{7}{3} \left(\frac{d}{L} \right)^2, \quad \Phi = \frac{12EI}{L^2 GkA} = 3 \left(\frac{d}{L} \right)^2$$

TABLE 4
RELATIONSHIP BETWEEN SPAN/DEPTH (L/d) RATIO AND THE NATURAL FREQUENCY RATIO, C_ω

Span/depth (L/d) Ratio	Frequency Ratio $C_\omega = \frac{\omega}{\omega_b}$			
	Clamped-Free		Clamped-Hinged	
	$\sqrt{1.01 + 0.75\Phi + (5.63 + 4.76\Phi)\lambda}$		$\sqrt{1.02 + 0.69\Phi + (9.75 + 46.28\Phi)\lambda}$	
	With RI	Without RI	With RI	Without RI
100	1.006	1.005	1.011	1.010
80	1.006	1.005	1.012	1.010
50	1.008	1.005	1.015	1.010
40	1.010	1.006	1.018	1.011
30	1.013	1.006	1.024	1.011
20	1.024	1.008	1.041	1.013
10	1.079	1.016	1.140	1.020
5	1.278	1.049	1.591	1.050
4	1.411	1.073	1.959	1.072

Legend:

RI = Rotary Inertia

ω = frequency from unified element solution

ω_b = frequency from exact (or classical solution)

The results of a comparative analysis of the effect of higher vibration modes using Timoshenko model and the proposed unified beam model for different support conditions are presented in Table 5. The unified element results are compared with the analytical results obtained by Leszek [15].

The following parameters were assumed [15]:

Modulus of elasticity, $E = 2.1 \times 10^{11}$ N/m²

Poisson's ratio, $\nu = 0.25$

Density of beam material, $\rho = 7800$ kg/m³

Shear correction factor, $k = 5/6$

Length of beam, $L = 1.0$ m

Width of beam section, $b = 0.02$ m

Depth of beam section, $d = 0.08$ m

TABLE 5
COMPUTED NATURAL FREQUENCIES OF CLAMPED-FREE BEAM

Mode of Vibration	Natural Frequency in rad/s				
	Unified Element Solution				Timoshenko Beam (Analytical Solution)
	With Shear Only	% Difference	With Shear and RI	% Difference	
1	159	0.00	159	0.00	159
2	995	0.10	994	0.00	994
3	2777	0.58	2769	0.29	2761
4	5412	1.16	5383	0.62	5350
5	8882	1.61	8806	0.74	8741

Legend:

RI = Rotary Inertia

Table 2 demonstrates that the effect of rotary inertia is to reduce the natural frequency of vibration of beams for all support conditions. The effect is more pronounced in deep beams compared to slender beams (about 6.7% and 0.07% reduction respectively for a clamped-free support).

The results in Table 3 show the accuracy of the unified element model. For a one-element mesh, the difference from the exact solution is in the order of 0.42% for a clamped-free support. It is also observed that the unified

element solution generally yields upper-bound values of the natural frequency.

Table 4 shows that, in general, the frequency ratio increases as the span-to-depth ratio decreases for all support conditions.

Table 5 shows that there is a convergence of the unified element and exact solutions at lower modes of vibration. The divergence becomes more pronounced at higher modes. Also the natural vibration frequency is less sensitive to rotary inertia at lower modes.

9 CONCLUSION

In this paper, the energy separation principle was successfully applied in the vibration analysis of Timoshenko beam. The proposed unified finite element model incorporates the effects of bending, shear deformation and rotary inertia by employing a bending-shear interaction factor. Explicit formulae for natural frequencies of Timoshenko beams based on the proposed unified element have been developed for different support conditions.

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The results show high accuracy and efficiency of the proposed unified element model in calculating the natural vibration frequencies of beams for different boundary conditions. The unified element solution generally yields upper-bound values of the natural frequency. Also, the frequency ratio increases as the span-to-depth ratio decreases. The results suggest that shear deformation contributes significantly to the natural vibration frequency for span-to-depth ratio less than 5, and should therefore be accounted for in design of such beams.

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